APPLICATION OF LEAST SQUARE METHOD AND CONJUGATE GRADIENT IN SOLVING SECOND ORDER LINEAR NONHOMOGENEOUS ODE

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ABSTRACT

Problem regarding second-order nonhomogeneous ordinary differential equations with Boundary Value Problem (BVP) commonly encountered in a wide range of fields and professions such as physics and engineering making them important to find a solution to solve the equations. They usually are solved using two theoretical methods which are known as undetermined coefficient and variation of parameters. Nevertheless, it is quite difficult and takes a lot of time to understand whenever involving a complicated equation. Researchers are more likely to use a numerical method in the form of least square method which is more practical and only requires a simple method to be understood compared to the theoretical method. In this research, there are three types of ordinary differential equation (ODE) problem that are chosen and solved by using both theoretical and least square method. Since the problem might come from the theoretical method, the functions are chosen based on the method. The three types of functions consist of exponential, algebraic and trigonometric. The least square method (LSM) cannot solve the equations by itself as there is inverse matrix comes from the system of linear equation which will lead to ill- conditioned matrix. To avoid such problems, Conjugate Gradient (CG) are applied. Then, the error of the equation is taken based on exact value and approximate methods to determine the best solution. From that, it shows that LSM can solve a second-order nonhomogeneous ordinary differential equation with BVP.

Keywords: Boundary value problem, ordinary differential equation, least square method, conjugate gradient

1.0 INTRODUCTION

One of the topics in the field of mathematics is differential equations. A Differential Equation seems to be an essential tool in a broad variety of applications including ordinary differential equations or partial differential equations [1]. The function usually represents physical quantities, the rates of change are represented by derivatives and the differential equation describes the relationship between them. When modeling the natural and physical world laws, the difference equation is useful. The differential equation, according to [2], is an equation covering one or more dependent variables in one or more independent variables. In any order, a differential equation may be in their order of differential. Through checking the order of the highest order derivatives in the equation the order can be decided. The general form of the second order differential equation can be seen as $ay'' + by' + cy = (t)$ [3]. The differential equation can be divided into two major separate forms, the Ordinary and Partial Differential Equations. Ordinary differential equation can be defined as a differential equation containing a single variable derivative. The differential equation can also be divided into two types, Linear Differential Equation (LDE) and Non-Linear [4]. Although both types are significant, in elementary applications linear differential equations are considered the most important.

2.0 LITERATURE REVIEW

System of Ordinary Differential Equation (ODE) is important as it frequently arises in mathematical models all over science and engineering [5]. Differential Equation (DE) is divided into two class types, linear and nonlinear. Linear DE is simple in mathematics and physics, while nonlinear means complex, as [1] claimed that linear DE is a 'friendly' DE family. As the solution of linear DE is highly advanced, linear DE is easy enough to solve, whereas non-linear DE is usually unsolved and is one of the topics of a lot of ongoing research.

Meanwhile, second order linear ODE usually appeared in many real-life situations. The general solutions of second order linear ordinary differential equation for the function y is

$$a2(x)y'' + a1(x)y' + a0(x)y = f(x) \qquad [1]$$

where $a2$, $a1$, $a0$ are constants. If $(x) = 0$, this equation is thought to be a second order homogenous linear ODE and if $f(x) \neq 0$, this equation is thought to be a second order non-homogenous linear ODE. In solving second order non-homogenous linear ODE, about two common methods that always been used which are undetermined coefficient (UC) and variation of parameter (VP) [6]. These theoretical methods provide the exact solution to non- homogeneous linear ODE of second order. The general solutions of second order non- homogenous linear ODE can also be stated in the form

$$\mathrm{y} = yc + yp \qquad [2]$$

where yc is a complementary solution for the homogenous solution of the non- homogenous solution. The term yp is a particular solution that satisfies the non- homogenous part which will be solved using UC and VP [7]. In addition, BVP is used for the estimation of ODE parameters. This is because [8] stated that there is a necessity to deduct parameters and initial condition for the trajectories to compute a solution for initial value problem (IVP). This can yield a problem in which the IVP can be difficult to overcome, or may have no solution at all, and can enter severely conditioned regions of the IVP, which can cause stability loss. The solution of BVP's second-order non homogeneous linear ODE is also important since many applications such as modeling of chemical reactions, thermal transfer and diffusion still arise in science and engineering.

According to [1] second order linear DE plays a significant role in many applications. Usually, differential equations described systems are so complex that purely analytical solutions are not manageable equations. At best, there are limited differential equations that can be solved analytically in a closed form [9]. Methods such as the use of complementary functions, particular integrals and variation of parameter methods exist for second and higher order, but there are very few kinds of equations that are solvable. A simple linear ODE of second order can be solved with various special functions such as Legendre. Nevertheless, the types of functions that must solve only fairly simple linear ODE are extremely complex after the second order. After years of research, the mathematician acknowledges that it is impossible to achieve explicit ODE solutions. Therefore, numerical method is used as another alternative method to solve the second order linear ODE [10].

According [3] shooting method is one of the usual numerical methods that researchers used in order to discover the solution of second order non-homogenous linear BVP for ODE. However [3] met the difficulties of not finding a solution algorithm for the general boundary values of problems. Most researchers have found another approach to overcome this issue by using a Least Square Model (LSM) nonhomogeneous linear BVP of ODE in the second order. In [11] research, He claimed that LSM is a feasible, simple, and easy-to-implement method since the BVP approach for second-order non-homogeneous linear ODE does not require an advance optimization method. Other authors such [11] also used LSM in order to approximate the solutions for the differential equation as approximation method gives more accurate result and minimum value of error when it is related with the theoretical method.

However, the final stage of solving LSM involves the system of linear equations $Ax = b$. That will contribute to the inverse matrix as well. The matrix is invertible only if the matrix is not singular. When the determinant is not zero, the matrix is called nonsingular matrix. The matrix is not invertible if the matrix is singular or nearly singular, that is if the matrix determinant is negative or zero. Singular and nonsingular matrix is called ill-conditioned matrix [12]. In order to avoid these problems, Conjugate Gradient method is applied.

In the research of [13] he said that CG is the most common method of solving large linear equation systems. CG method is useful when used to solve large systems with non-zero entries (singular) as the iterative approximation method. Such problems also often occur when solving BVPs.

3.0 PROBLEM STATEMENT

Solving a BVP linear non-homogeneous second-order ODE is very important, as it often occurs in different fields and professions, especially engineering, biology and research. In-circumstances of real life the DE or ODE models that make the problem very difficult to solve [1] are not good enough as they have limited capability of finding the solutions [11]. It can be difficult to solve the UC and VP integration problem and the final solution is not confirmed. Such methods are often very complicated and time-consuming. The complexity of models of physical, chemical and biological laws frequently contributes to differential equations that cannot be overcome in a closed analysis. Therefore, the other method has been used which is by using numerical methods [14].

Numerical methods are decent alternative methods in solving DE [1]. Normally researchers use this method to approximate results and to provide more precise error data. Many of the methods are shooting, Runge-Kutta, finite differences and Euler 's method. Nevertheless, they are both restricted and efficient in solving the differential equation. Through this thesis, the least square method is chosen as the numerical method, which is simpler to understand and does not require repetitive calculations in the form of an iterative method.

Nevertheless, at the end of the method, this method relates to the inverse matrix and is potentially a problem because the matrix is not invertible. The matrix is invertible only if the matrix is not singular. The non-zero matrix is referred to as non-singular. The inversion process cannot be used on a matrix if the matrix is singular or nearly singular which is if the determinant of the matrix is zero or close to zero [15]. Therefore, Conjugate Gradient method is required in order to evade these problems.

4.0 SIGNIFICANCE OF RESEARCH

Second-order linear ODE offers a variety of science and engineering applications such as heat transfer and chemical diffusion. The need for simple technology is widespread in science with the quickly evolving fields of physics, chemistry and biology. The absences of this project will help the researchers to solve the ODE problem using simple methods other than the use of numerous theoretical ODE methods. In addition, error analysis determines the reliability of these methods, where the lowest error percentage as the most accurate solution is measured. Therefore, researchers can decide which method is the best method to be used without attempting them one by one in their future study. This approach may also help in the solving of complex differential equations.

5.0 RESEARCH METHODOLOGY

The research steps are clearly described one by one in the form of one step each. Therefore, the research methodology shown clearly by using the flowchart shows in Figure 1 below:

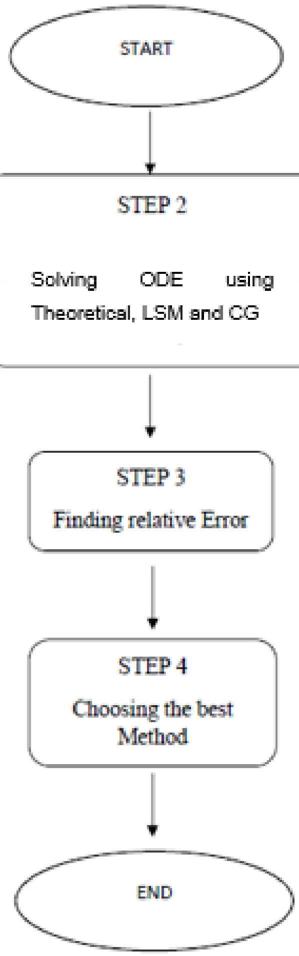

Figure 1: Flowchart of the Research Method

6.0 RESULTS AND DISCUSSION

Different types of second order ODE were selected in order to clarify the behaviors of the method. Trigonometric, algebraic and exponential types are selected. Apart from that, the nonhomogeneous ODE with BVP will be used as the guideline in this research.

Table 1: List of Functions

No.	Function	BVP
1.	$y'' + 3y' + 2y = x^2$	$y(0) = 0, y(1) = 0$
2.	$y'' - 2y' = \sin(4x)$	$y(0) = 0, y(1) = 1$
3.	$y'' + 2y' + y = xe^{-x}$	$y(0) = 1, y(1) = 0$

From the calculations of the three functions from the Table 1 above, the researcher got the results and plot into the graph in order to determine the comparison between the theoretical method, least square method and the conjugate gradient method.

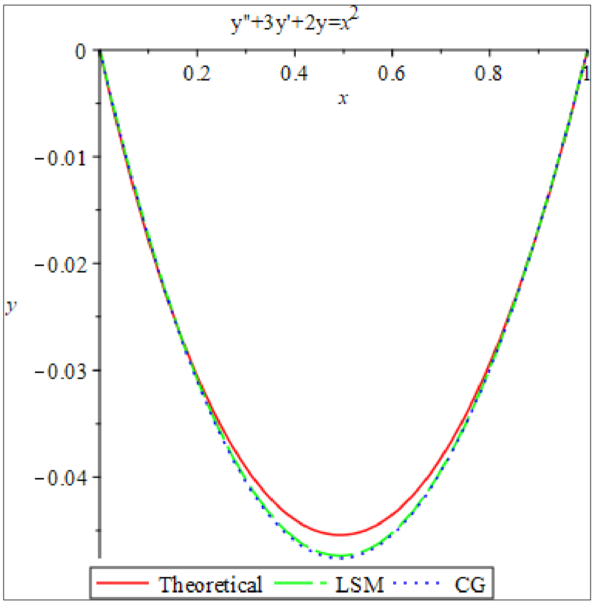

Figure 2: $y'' + 3y' + 2y = x^2$, BVP: (0) = 0, (1) = 0

Figure 2 shows that the graph from the first ODE function was plotted in one figure. From the graph, the researcher fined that LSM and CG were almost lay in the same line but LSM is quite near to theoretical method.

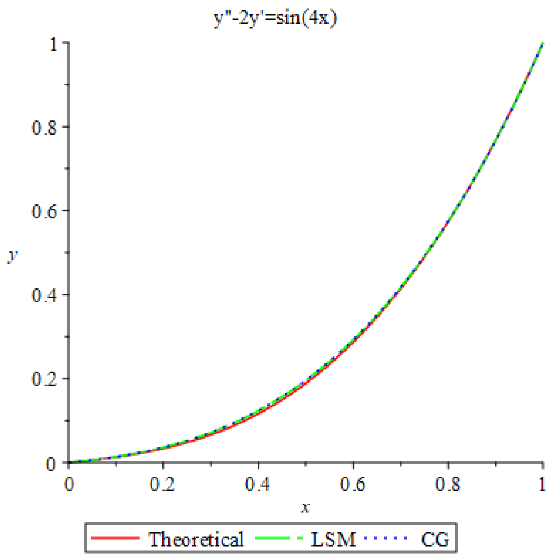

Figure 3: $y'' - 2y' = \sin(4x)$, BVP: (0) = 0, $y(1) = 1$

Figure 3 shows that the graph from the second ODE function was plotted in one figure. From the graph, the researcher fined that LSM and CG were almost lay in the same line with the theoretical method.

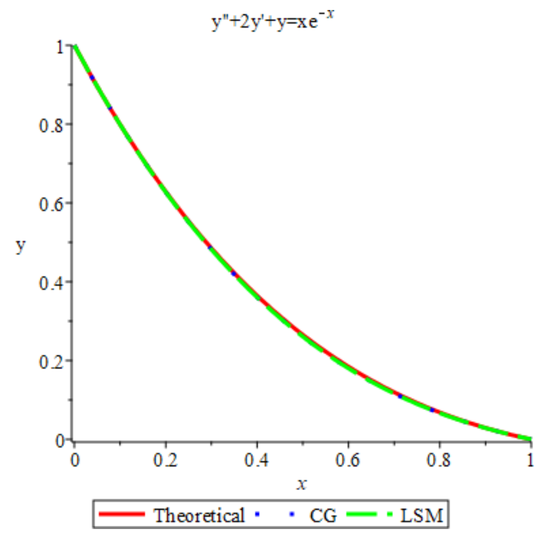

Figure 4: $y'' + 2y' + y = xe^{-x}$, BVP: (0) = 1, (1) = 0

Figure 4 above shows the graph from the third ODE function. From the figure, the researcher fined that theoretical method, LSM and CG method also almost lie in the same line.

Finally, based on the results of theoretical, LSM and CG method, the researcher fined the percentage error between the numerical method and theoretical method. The ODE solution will represent the exact value when the approximate value is obtained by the LSM and CG method. So, the method to find the error is calculated by using a percentage error.

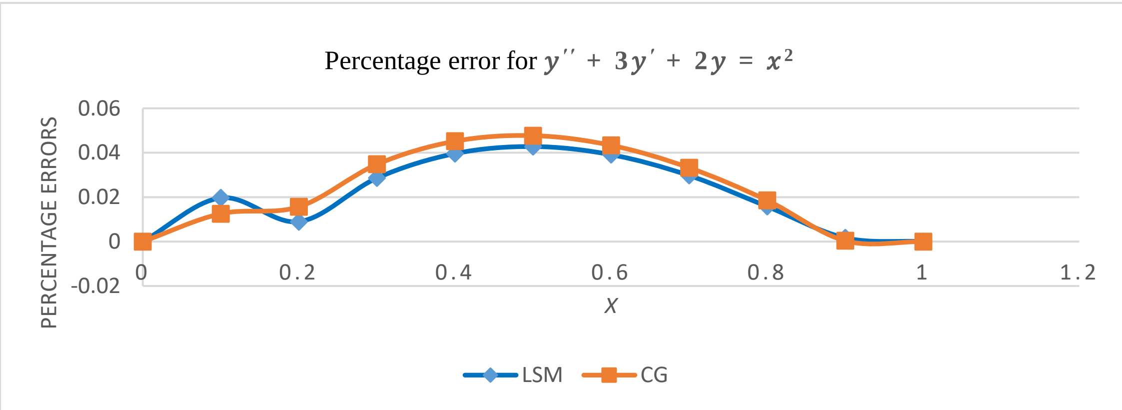

Figure 5: Percentage error for $y'' + 3y' + 2y = x2$, BVP: $y(0) = 0$, $y(1) = 0$

Figure 5 shows that the percentage error between two numerical method which are LSM and CG method were plotted in one figure in order to find the lowest percentage error. Based on the figure, the researcher find that the LSM was the lowest error compared to CG for the solution of first function ODE.

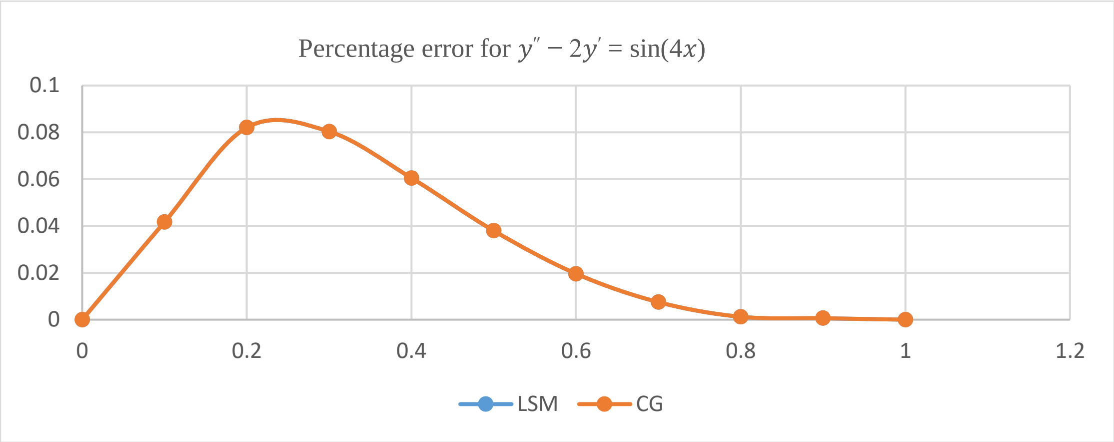

Figure 6: Percentage error for $y'' - 2y' = \sin(4x)$, BVP: $y(0) = 0$, $y(1) = 1$

Figure 6 shows that the percentage error between two numerical method which are LSM and CG method were plotted in one figure in order to find the lowest percentage error. Based on the figure, the researcher find that the LSM and CG were in the same line of the percentage error for the solution of second function ODE.

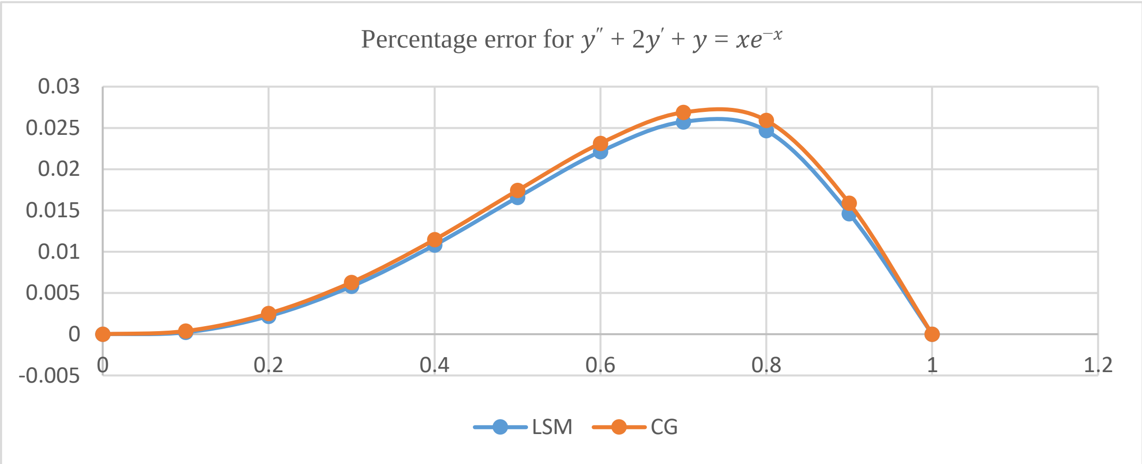

Figure 7: Percentage error for $y'' + 2y' + y = xe^{-x}$, BVP: $(0) = 1$, $(1) = 0$

Figure 7 shows that the percentage error between two numerical method which are LSM and CG method were plotted in one figure in order to find the lowest percentage error. Based on the figure, the researcher find that the LSM was the lowest error compared to CG for the solution of third function ODE.

All three methods can be used to solve the three functions of ODE based on the results obtained. However, by interpolating all three results from each method and also the actual value in the same graph, the value obtained from conjugate gradient method is the furthest from the actual value. In addition, based on Figure 5 and Figure 7 the CG method has the largest percentage error and less accuracy compared to LSM that used in this study after calculating based on calculating the percentage error for all three methods. Therefore, the best method for solving ODE determined by calculation the percentage error. The LSM has a smaller percentage error based on the figures above. It is proven that LSM is the best method to be used to solve the ordinary differential equations.

7.0 CONCLUSION AND RECOMMENDATION

From the error analysis, data obtained from the LSM has the smallest error and from the graph plotted the data is closer to the data compared to data obtained by using LSM and CG. According to the data analysis for all three methods, the CG method has the biggest percentage error compared to LSM, making it the least relevant method compared to other methods that are used to calculate the percentage error for solving the ODE. The data acquired using the CG method is the furthest from the original data, as shown in the graph.

In conclusion, the least square method, when compared to the conjugate gradient method, is the best method for solving ordinary differential equations since it has the smallest percentage error and the data collected is the closest to the actual data.

For recommendation, only linear differential equation was tested by all methods. Researchers may extend the research field in future to solve nonlinear differential or partial differential equations. Otherwise, researchers could also use quartic or quantic forms despite the cubic shape to test LSM's ability to solve the higher form of differential equation.

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