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NUMERICAL STUDIES FOR SOLVING ABEL'S DIFFERENTIAL EQUATION

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ABSTRACT

This paper uses the Abel's Equation, often used in various fields including physics and engineering, represents a mathematical model that can be complex to solve analytically. This research focuses on solving the Abel's Equation using several numerical methods: Euler's method, Taylor series method, Adomian decomposition method, and Runge-Kutta method. Each method has its advantages and applicability depending on the specific characteristics of the equation being solved.

1.0 Introduction

One of the most important first-order nonlinear ordinary differential equations is known as Abel's equation of the first kind, this equation often appears in multiple applications such as fluid dynamics, heat transfer, and mathematical biology [10-13]. It is solved in the following ways:

Euler's Method: Euler's method is a straightforward numerical technique to solve differential equations through approximation the solution using a linear approximation. For Abel's equation, the method involves discretizing the interval and iteratively updating the solution. While simple and easy to implement, Euler's method may suffer from stability and accuracy issues, especially for stiff equations or when a high precision is required [14].

Taylor Series Expansion: The Taylor series method provides an analytical approach by expanding the solution in a series around a point. For Abel's equation, the solution can be expressed as a Taylor series and computed term-by-term. This method offers high accuracy for smooth functions but may require extensive computation for higher-order terms and can be cumbersome for complex boundary conditions [22, 25 and 26].

Adomian Decomposition Method: The Adomian decomposition method (ADM) is an analytical technique that decomposes the solution into a series of functions. For Abel's equation, ADM involves breaking the problem into simpler sub-problems and solving iteratively. This method is advantageous for handling nonlinear terms and provides an approximate solution that converges quickly with fewer terms compared to other analytical methods [1- 4].

Runge-Kutta Method: The Runge-Kutta methods are a set of iterative techniques used to solve ordinary differential equations, characterized by higher accuracy compared to the Euler method. Among these methods, the fourth-order Runge-Kutta method is widely used due to its balance between accuracy and computational effort. This method relies on estimating the solution by evaluating the slope at several points within each interval, providing a highly accurate numerical solution to Abel's equation [9, 18].

1.1 Definition:

The first kind Abel's differential equation takes the form [16, 17]

$$\frac{dy}{dx} = M(x) + S(x)y + R(x)y^2 + T(x)\, y^3 \tag{1}$$

Where $M(x),\ S(x),\ R(x)$ and $T(x)$ are arbitrary function of x. Eq. (1) must be solved together with the initial condition $y(x) = y_0$.

2.0 Analysis of Methods

2.1 Euler Method: [15]

Euler method is one of methods for solving an initial value problem (IVP) of the form

$$\frac{dy}{dx} = f(x, y), \quad a \le x \le b, \quad y(a) = y_0 \tag{2}$$

Although it is not very accurate, its simplicity may make it difficult to use for the initial value problem of Apple. We need to determine the values at a specific set of points. We have to determine values of w_i at discrete set of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

The approximate solution for w_i is represented here $y_i = y(x_i)$. For simplicity, the approximate solution will be sought at equally spaced points.

$x_i = a + ih$, where $i = 1,2, \dots, n$

$h = \frac{b-a}{n}$, For some positive integer n

Now, Taylor's theorem is used to deduce Euler's method $y(x)$, is the characteristic solution of equation (1) gives continuous derivatives.$[a, b]$, so that $i = 1,2, \dots, n-1$

$$y(x_{i+1}) = y(x_i) + (x_{i+1} - x_i)y'(x_i) + \frac{(x_{i+1}-x_i)^2}{2}y''(\xi_i) \tag{3}$$

For some numbers ξ_i in $(x_{i+1} - x_i)$. so that $h = x_{i+1} - x_i$it is

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\xi_i) \tag{4}$$

And $y(x)$ the differential equation (1) is satisfied.

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2}y''(\xi_i) \tag{5}$$

Euler's method arises by dropping the error term and replacing y_i (exact solution) by w_i (approximate solution): $w_0 = a$

$$w_{i+1} = w_i + h\, f(x_i, y_i), \quad \text{for each} \quad i = 1,2, \dots, n-1 \tag{6}$$

2.2 Taylor Expansion Method: [23, 24]

Taylor's Theorem: Suppose y is continuous on the closed interval $[a, b]$ and has $n+1$ continuous derivatives on open interval (a, b). If x and x_0 are points in (a, b), then the Taylor series expansion of $y(x)$ about x_0 **:**

$$y(x_0) + y'(x_0)(x - x_0) + \frac{(x - x_0)^2}{2!}y^{(2)}(x_0) + \frac{(x - x_0)^3}{3!}y^{(3)}(x_0) + \cdots$$

or

$$y(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} y^{(n)}(x_0), \qquad n, x_0 \in (a, b) \tag{7}$$

where $y(x)$ is convergence.

2.3 Adomian Decomposition Method: [5 – 8]

A differential equation can be written as

$$Fy = g \tag{8}$$

where F it is a nonlinear differential operator that includes linear and nonlinear boundaries.. The linear part of F is

$$F = L + R \tag{9}$$

where L is an invertible operator and R is a linear operator's residual. Using L as the highestorder derivative simplifies the equation to

$$Ly + Ry + Ny = g \tag{10}$$

where Ny corresponds to the non-linear terms. Rewrite (10), we get

$$Ly = g - Ry - Ny \tag{11}$$

and by application L^{-1} to the equation (11), becomes

$$L^{-1}\ (Ly) = L^{-1}g - L^{-1}(Ry) - L^{-1}(Ny) \tag{12}$$

Suppose h it is the solution of the homogeneous equation $Ly = 0$, with the given initial boundary conditions. Then the general solution of equation (12) is,

$$y(x) = h + L^{-1}g - L^{-1}(Ry) - L^{-1}(Ny) \tag{13}$$

The following problem pertains to the analysis of the nonlinear limit Ny. Adomian developed a very elegant method, where the approximate solution to equation (13) can be written as follows:

$$y(x) = y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots = \sum_{n=0}^{\infty} \lambda^n y_n(x) \tag{14}$$

where λ is constant, whereas $y_0(x),\ y_1(x),\ y_2(x), \ldots, y_n(x)$ are sought. If the non-linear operator $N$is attempted to equation (14)

$$Ny = N(y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots).$$

The nonlinear limit Ny will be analyzed through the infinite Adamian series.

$$Ny = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots = \sum_{n=0}^{\infty} \lambda^n A_n, \tag{15}$$

with

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n}\lambda^n y_k(x)\right)\right] \tag{16}$$

where A_n are called Adomian polynomials, if $N(y) = f(y)$, the Adomian polynomials are given as:

$$A_0 = f(y_0) \qquad A_2 = \frac{y_1}{2!}f''(y_0) + y_2 f'(y_0)$$
$$A_1 = y_1 f'(y_0)$$

$$A_3 = \frac{{y_1}^3}{2!}f'''(y_0) + y_1 y_2 f''(y_0) + y_3 f'(y_0)$$

$$\vdots$$

Suppose $L^{-1}Ry$, $L^{-1}Ny$ have a λ order, then Equation (13) can be written as

$$y(x) = h + L^{-1}g - \lambda L^{-1}(Ry) - \lambda L^{-1}(Ny). \tag{17}$$

Put equation (14) and (15) in equation (17), then we obtain

$$\sum_{n=0}^{\infty}\lambda^n y_n(x) = h + L^{-1}g - \lambda L^{-1}\left(R\left(\sum_{n=0}^{\infty}\lambda^n y_n(x)\right)\right) - \lambda L^{-1}\left(\sum_{n=0}^{\infty}\lambda^n A_n\right) \tag{18}$$

By equating the coefficients of equal powers λ on both sides of equation (18), we obtain:

$$y_0 = h + L^{-1}g$$
$$y_1 = -L^{-1}R(y_0) - L^{-1}(A_0)$$
$$y_2 = -L^{-1}R(y_1) - L^{-1}(A_1)$$
$$\vdots$$

It can generally be expressed through recursive relations.

$$y_n = -L^{-1}R(y_{n-1}) - L^{-1}(A_{n-1}), \qquad n \geq 1 \tag{19}$$

Then, the approximate solution is given by

$$y = y_0 + y_1 + y_2 + y_3 + y_4 + \cdots \tag{20}$$

2.4 Runge-Kutta 4th Method: [19, 20, 21]

Accuracy is achieved through the use of Runge-Kutta methods, which eliminate the need for derivatives by evaluating the function $f(x, y)$ at a specific point in each subinterval. The fourth method of the Runge-Kutta methods, used in initial value problems, is considered the most accurate among common methods, as it requires several analytical steps. Runge-Kutte 4th method is

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (21)$$

And for step size $h = x_{i+1} - x_i$

Where the coefficients are

$$k_1 = h\, f(x_i, y_i)$$

$$k_2 = h\, f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right)$$

$$k_3 = h\, f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2\right)$$

$$k_4 = h\, f(x_i + h, y_i + k_3) \qquad (22)$$

3.0 Numerical Results:

This method is considered a reliable method that requires less effort compared to traditional procedures. To better illustrate the idea, we show some examples and MATLAB is used to calculate all the results

3.1 Example: Consider the Abel's differential equation

$$\frac{dy}{dx} = x + 3xy + 3xy^2 + x\,y^3 \qquad (23)$$

with initial condition $y(0) = 0,$ or $y(0) = -2.$ Its general solution is given by

$$y(x) = -1 \pm \frac{1}{\sqrt{1-x^2}} \qquad (24)$$

Table1: The scheduling of Euler, Taylor, Adomian, fourth-order RK methods, and Exact Solution for Example on h=0.1

h	x	Exact	Euler	Taylor	Adomian	RK th4
0.1	0.1	5.037815e-03	0	5.000000e-03	5.037813e-03	5.037846e-03

0.1	0.2	2.062072e-02	1.000000e-02	2.022612e-02	2.062036e-02	2.062085e-02
0.1	0.3	4.828483e-02	3.060602e-02	4.677397e-02	4.827511e-02	4.828518e-02
0.1	0.4	9.108945e-02	6.344572e-02	8.691854e-02	9.098434e-02	9.109023e-02
0.1	0.5	1.547005e-01	1.115524e-01	1.447019e-01	1.539916e-01	1.547022e-01
0.1	0.6	2.500000e-01	1.802213e-01	2.271993e-01	2.463266e-01	2.500036e-01
0.1	0.7	4.002800e-01	2.788587e-01	3.473313e-01	3.836566e-01	4.002868e-01
0.1	0.8	6.666666e-01	4.252670e-01	5.307673e-01	5.935570e-01	6.666543e-01
0.1	0.9	1.294157e+00	6.568885e-01	8.356596e-01	9.249986e-01	1.292797e+00

Table2: Tabulation Euler, Taylor, Adomian, and RK th4 error methods with the Exact solution for Example on h=0.1

h	x	Exact	Abs. Error Euler	Abs. Error Taylor	Abs. Error Adomian	Abs. Error RK th4
0.1	0.1	5.037815e-03	1.000000e+00	7.506281e-03	2.735237e-07	6.170491e-06
0.1	0.2	2.062072e-02	5.150510e-01	1.913605e-02	1.754349e-05	6.445162e-06
0.1	0.3	4.828483e-02	3.661359e-01	3.129059e-02	2.013826e-04	7.216313e-06
0.1	0.4	9.108945e-02	3.034788e-01	4.578913e-02	1.153872e-03	8.628848e-06
0.1	0.5	1.547005e-01	2.789134e-01	6.463157e-02	4.582008e-03	1.096593e-05
0.1	0.6	2.500000e-01	2.791146e-01	9.120259e-02	1.469337e-02	1.445700e-05
0.1	0.7	4.002800e-01	3.033409e-01	1.322792e-01	4.152941e-02	1.686821e-05
0.1	0.8	6.666666e-01	3.620993e-01	2.038490e-01	1.096643e-01	1.851733e-05
0.1	0.9	1.294157e+00	4.924198e-01	3.5428280e-01	2.852502e-01	1.050690e-03

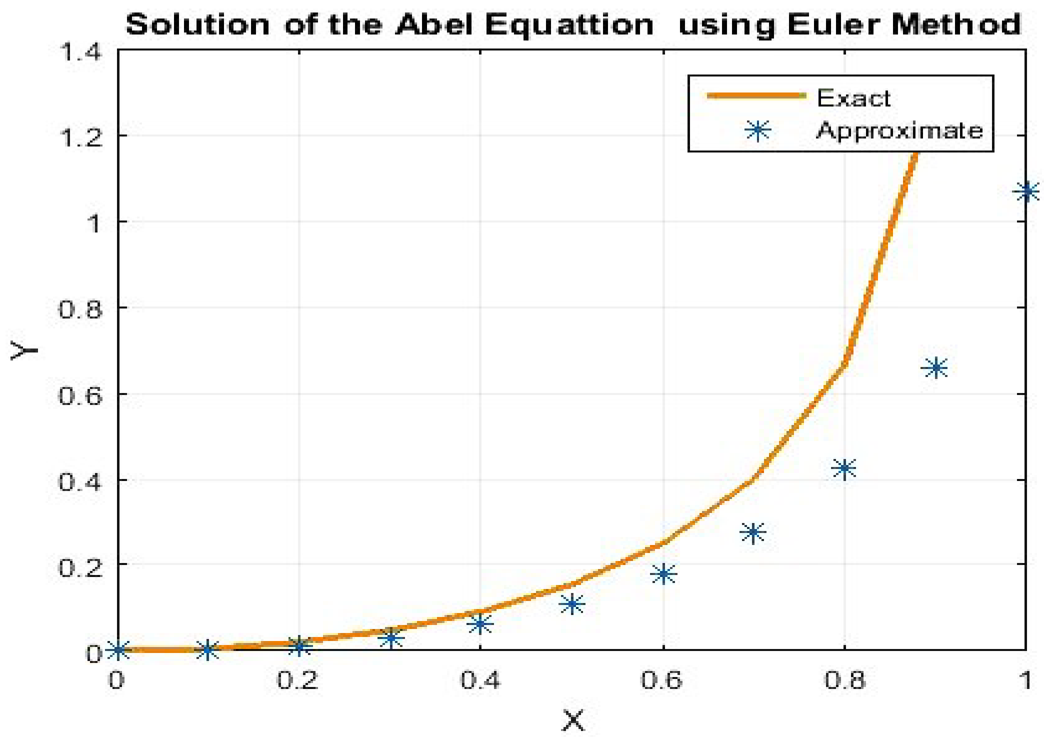

Fig. 1: Compare of Approximate Solution and Exact Solution for Example

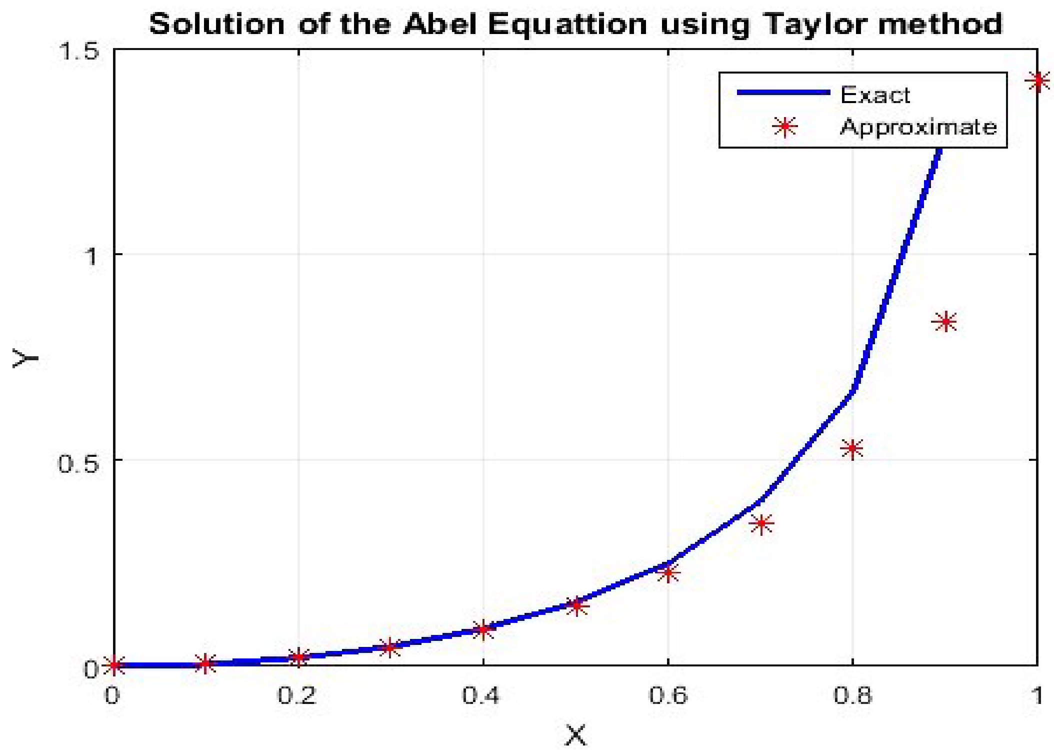

Fig. 2: Compare of Approximate Solution and Exact Solution for Example

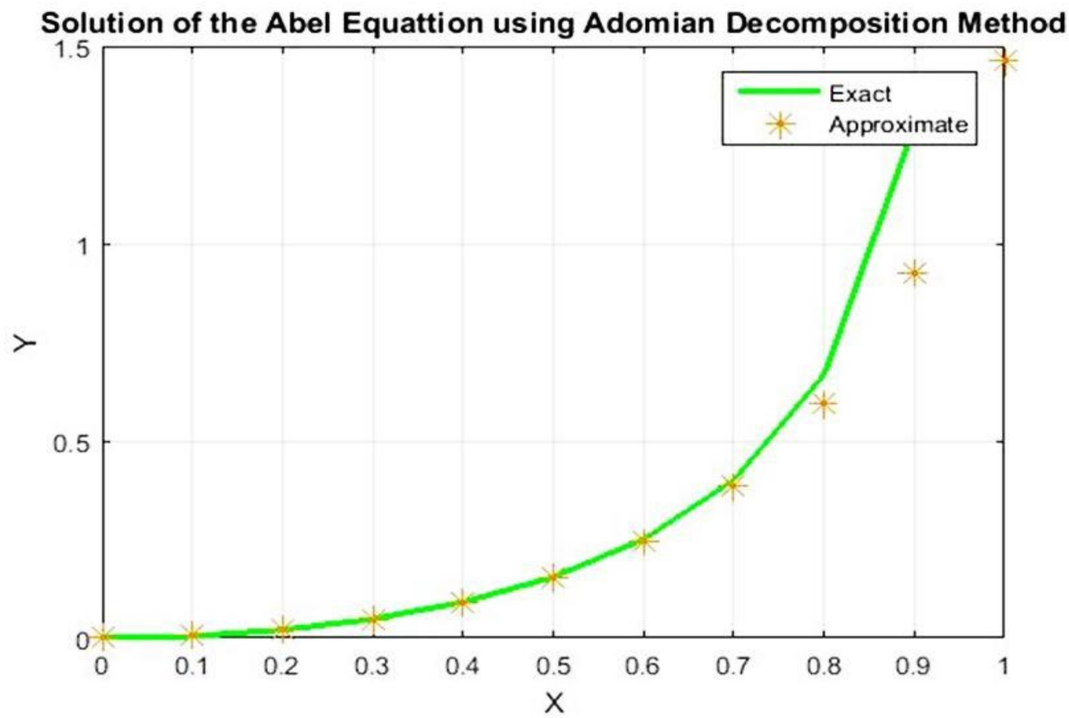

Fig. 3: Compare of Approximate Solution and Exact Solution for Example

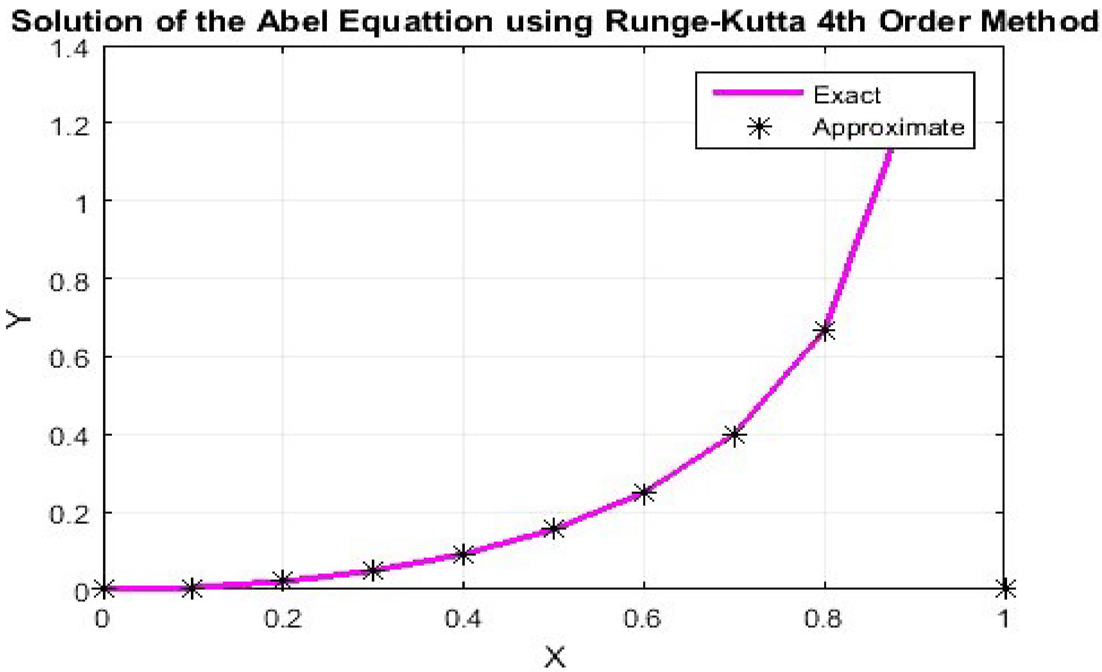

Fig. 4: Compare of Approximate Solution and Exact Solution for Example

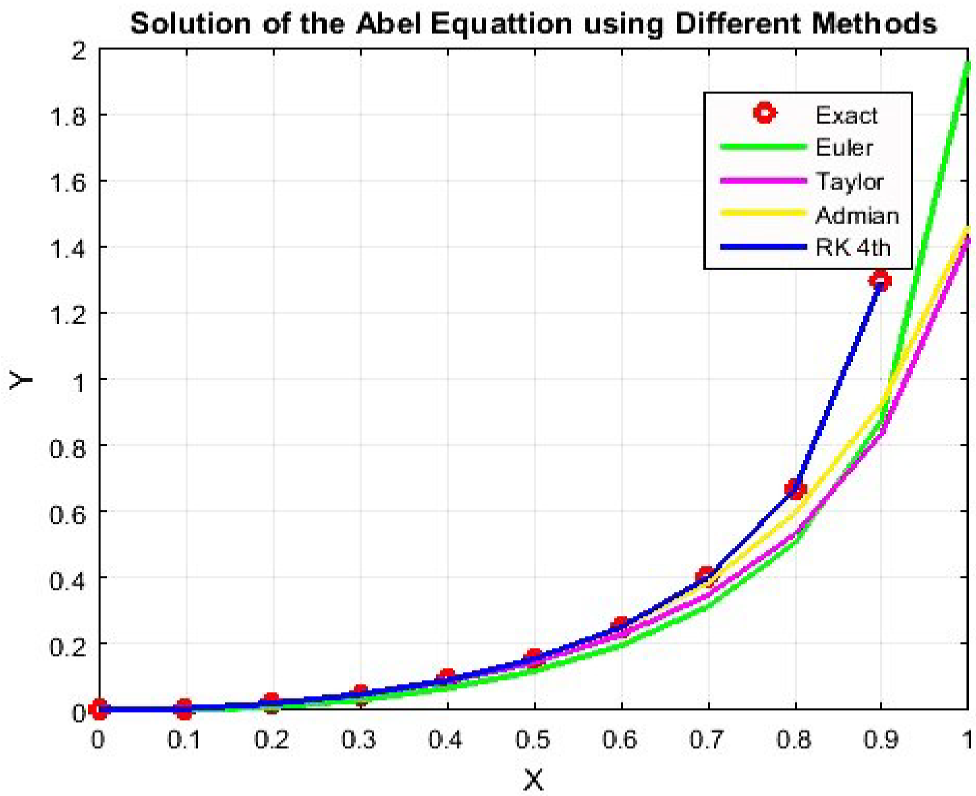

Fig. 5: Compare Approximate Solutions and Exact Solution for Example

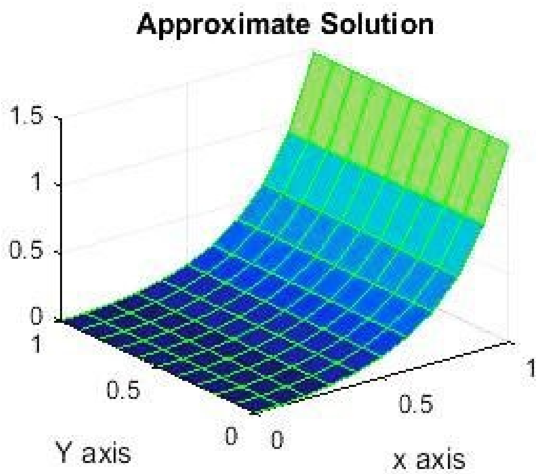

Fig. 6: Approximate Solution

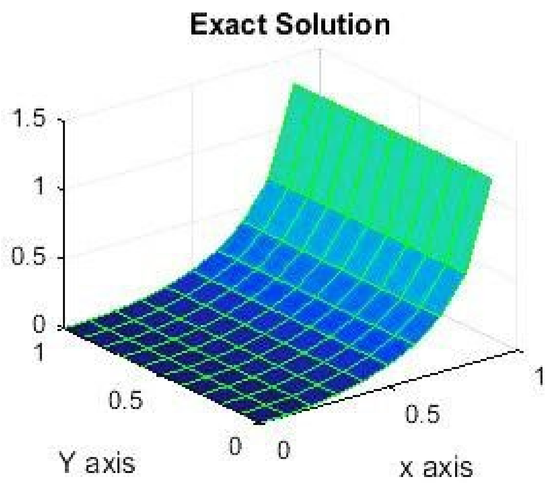

Fig. 7: Exact Solution

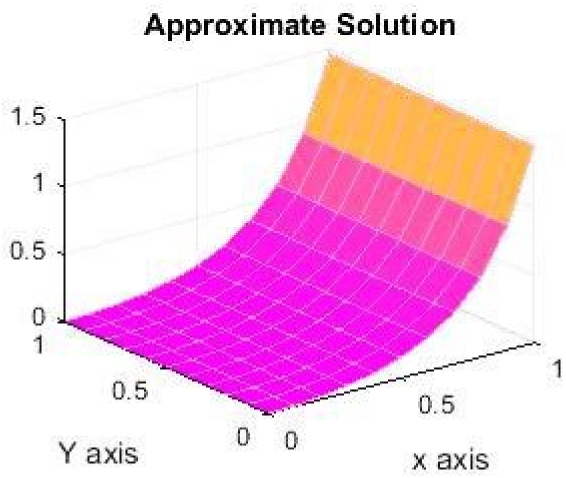

Fig. 8: Approximate Solution

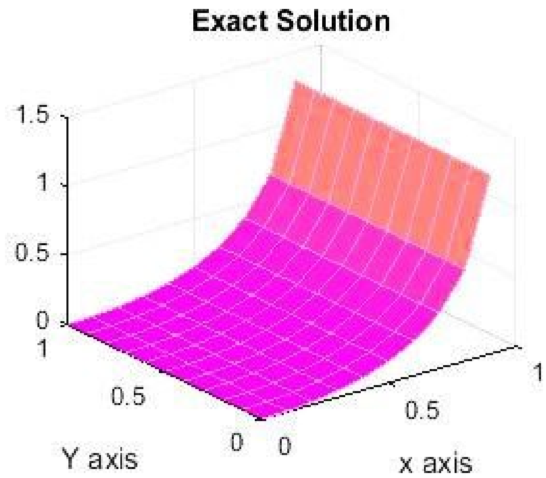

Fig. 9: Exact Solution

4.0 Conclusion

Each method offers unique advantages for solving Abel's differential equation. Euler's method is simple but less accurate, Taylor series method provides high accuracy with small step sizes, Adomian decomposition is effective for complex nonlinear equations, and RungeKutta methods strike a good balance between accuracy and computational cost. The choice of method depends on the specific requirements of the problem, including the desired accuracy and computational resources.

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